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Rapidly-increasing Special Functions

The study of special functions involving factorials and exponentials has spanned more than three hundred years and this has contributed to advances in computer science in fields such as computability, algorithm analysis [14] and tractability.

Complete Graphs K_n

In graph theory, a graph can be defined as an algebraic structure comprising two sets V , a set of vertices (or nodes) and E , a set of edges (or arcs) [3]. The simplest graphs, in which V and E are finite sets, can be represented in the form of a picture similar to the one shown in figure 1.

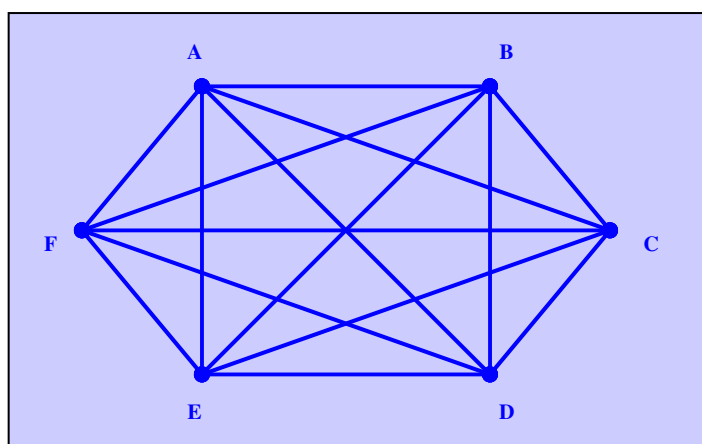


Figure 1. A picture of complete graph K_6 .

In figure 1, $V = \{A,B,C,D,E,F\}$ and E comprises all of the edges needed to completely connect each of the six vertices. In this case, the graph is completed using fifteen edges and can be viewed as a special case of K_n for which $n=6$. Computer engineers spend many hours seeking paths through graphs satisfying various properties [2,4,5] and two problems are particularly interesting.

One important problem seeks a path starting at a given vertex and ending at the same vertex having passed through every other vertex precisely once. This is called a Hamilton circuit, named after Hamilton. So for instance, we can see from figure 1 that, starting at vertex A, there are 120 distinct Hamilton circuits in K_6 . In general, there are $\Gamma(n) (= (n-1)!)$ distinct Hamilton circuits in K_n . This would have immediate application, say, in a computer network comprising n completely connected computers where, working from a given node, every other node might need to be tested exactly once [7].

Another problem seeks a path starting at a given vertex and ending at the same vertex having passed through every edge precisely once [11, 12]. This is called an Euler circuit, named after Leonhard Euler. We shall see later that, in general, the number of distinct Euler circuits in K_n can be calculated using a function $D(n)$. This would also have application, say, in a computer network comprising n completely connected computers in which, working from a given node, every connection might need to be tested exactly once.

Euler’s Gamma function $\Gamma(x)$

Leonhard Euler, a Swiss mathematician, was born in 1707 and was one of the greatest and most prolific mathematicians of the 18th century [9,15]. He wrote more than eight hundred papers, many of which were published posthumously, mainly in St Petersburg and Berlin. His attempts to explain the nature of functions and his success in the study of infinite series led his successors, notably Abel and Cauchy, to introduce ideas of convergence and rigorous argument into mathematics.

Euler’s *Introductio in Analysin Infinitorum* (1748) provided the foundations of analysis. He showed that any complex number to a complex power can be written as a complex number. He worked on the number e , the base of natural logarithms, which is known as Euler’s number. He gave that number its symbol, and introduced the symbol i , for the square root of -1 and also the notation $f(x)$, for a function of the variable x .

Euler’s gamma function, $\Gamma(x)$, originated by Leonard Euler around 1730, was a function of one variable defined by the integral

$$\int_0^{\infty} e^{-t} t^{x-1} dt, \tag{1}$$

for strictly positive integer values of x and later extended to strictly positive real values [16].

In fact, the gamma function for strictly positive values of x satisfies the recurrence relation:

$$\Gamma(x + 1) = x \Gamma(x) \text{ with } \Gamma(1) = 1. \tag{2}$$

In the particular case where x is a positive integer n , $\Gamma(n)$ is referred to as the factorial function:

$$\Gamma(n) = (n-1)! \tag{3}$$

Furthermore, equation (2) can be used to generalise the gamma function for negative values of x in a process of analytical continuation, although this will be the subject of a separate paper.

The following properties follow from various substitutions in (1) and recurrence relation (2):

$$\Gamma(1/2) = \sqrt{\pi} \tag{4}$$

$$\Gamma(n + 1/2) = \frac{(2n)! \sqrt{\pi}}{(n)! (4)^n} \text{ for } n \geq 1 \tag{5}$$

A graphical representation of $\Gamma(x)$ based on properties (1) – (5), is shown in figure 2.

The graph is continuous and positive on $(0, \infty)$ and, in this interval, possesses continuous derivatives of all orders n given by [16]

$$\Gamma^n(x) = \int_0^\infty e^{-t} t^{x-1} (\ln t)^n dt, \quad (6)$$

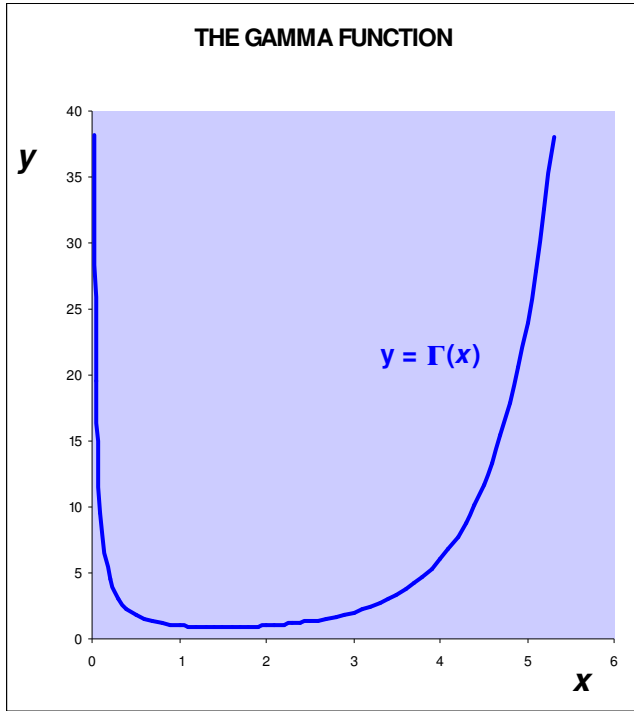


Figure 2. A graphical representation of $\Gamma(x)$.

For any rapidly-increasing function of x it is often useful to know an approximation for large values of x [6,10]. In the case of $\Gamma(x)$, Stirling¹ introduced the following result [1,17]:

$$\text{As } x \rightarrow \infty, \Gamma(x+1) \sim \sqrt{2\pi} x^{x+1/2} e^{-x} \quad (7)$$

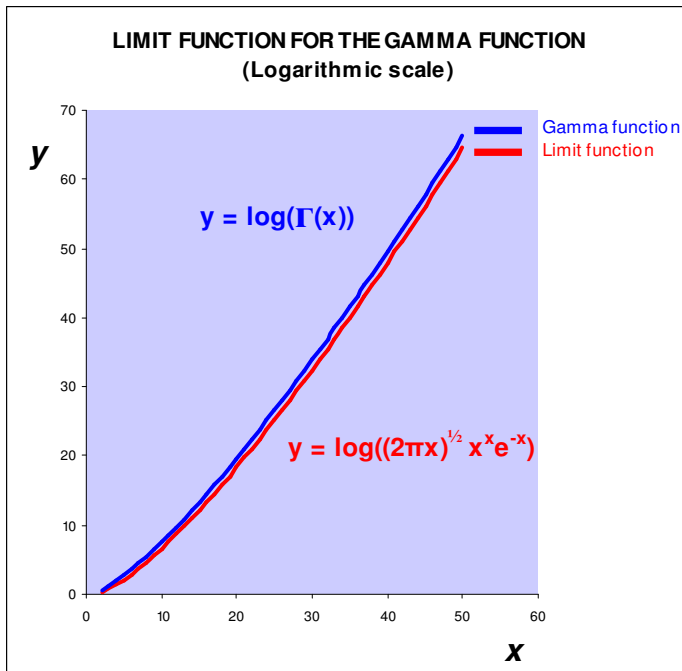


Figure 3. A limit function for the gamma function, $\Gamma(x)$.

1. In fact, the result was first introduced by de Moivre and Stirling later showed the value of the coefficient to be $\sqrt{2\pi}$.

The Beta function $B(x,y)$

The Beta function, also introduced by Euler around 1730, is a function of two variables that can be defined by the by the integral

$$\int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad (8)$$

for positive integer values of x and y and later extended to positive real values.

In fact, the Beta function is related to the Gamma function by

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \quad (9)$$

In the particular case where x and y are strictly positive integers m and n , respectively, $B(m, n)$ can be expressed in factorial terms as:

$$B(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}. \quad (10)$$

The following properties follow from various substitutions in (8) – (10):

$$B(x, y) = B(y, x) \quad (11)$$

$$B(x, y) = \int_0^\infty u^{(x-1)} (1+u)^{-(x+y)} du, \quad (12)$$

$$B(x+1, y+1) = \frac{xy}{(x+y)} B(x, y) \quad (13)$$

Note that using equation (8) and equation (9) with $x = y = 1/2$ yields

$$[\Gamma(1/2)]^2 = \int_0^1 t^{x-1} (1-t)^{y-1} dt,$$

which, following a substitution $t = \sin^2 \theta$, gives

$$[\Gamma(1/2)]^2 = 2 \int_0^{\pi/2} d\theta = \pi,$$

confirming result (4).

A graphical representation of $B(x,y)$, based on properties (8) – (13), is shown in figure 4. The graph is continuous and positive on $(0, \infty)$.

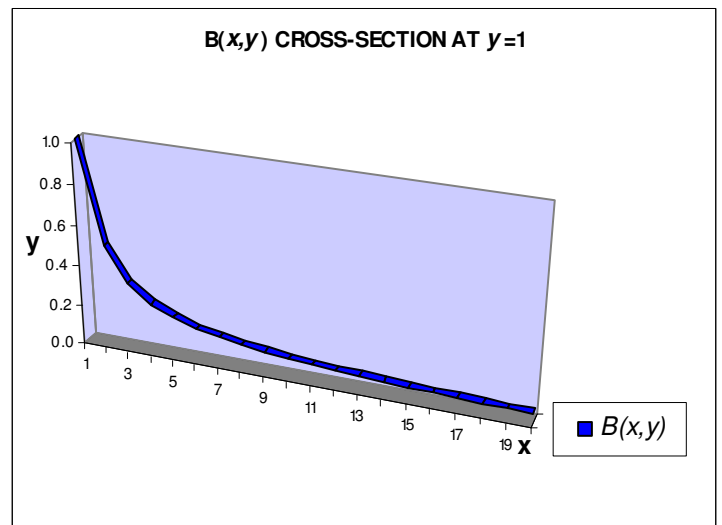


Figure 4. A cross-section of the Beta function, $B(x, 1)$.

A graph of the reciprocal function $B^{-1}(x,y)$ is shown in figure 5.

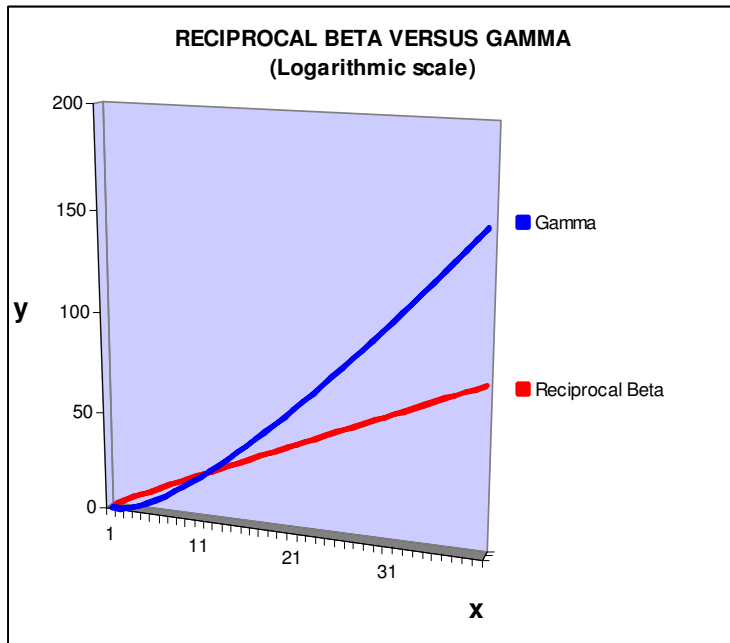


Figure 5. The reciprocal Beta function $B^{-1}(x,x)$ versus $\Gamma(x)$. The least integer value for which $B^{-1}(x,x) < \Gamma(x)$ is 13.

Clearly, $B^{-1}(x,y)$ is a rapidly-increasing function, though not as rapidly increasing as $\Gamma(x)$ [14], and one which can be defined as

$$B^{-1}(x, y) = \frac{\Gamma(x+y)}{\Gamma(x)\Gamma(y)}. \quad (14)$$

Properties which readily follow from (11) and (13) are:

$$B^{-1}(x, y) = B^{-1}(y, x) \quad (15)$$

$$B^{-1}(x+1, y+1) = \frac{x+y}{(xy)} B(x, y) \quad (16)$$

The $d(x)$ function

Consider a function of one variable $d(x)$ defined by the integral

$$\frac{2^{(x+1)}}{\sqrt{\pi}} \int_0^\infty e^{-t} t^{x+1/2} dt, \quad (17)$$

for positive real values of x . It can be shown that a closed form expression is

$$d(x) = \frac{(2x+1)!}{2^x(x)!}. \quad (18)$$

For integer values, we can see that the value of $d(n)$ can be written generally as:

$$(2n+1)(2n-1)(2n-3) \dots 5.3.1 \quad (19)$$

so that, for examples, $d(2) = 15$, $d(3) = 105$ and $d(4) = 945$.

In fact, $d(x)$ for $x \geq 0$ satisfies the recurrence relation:

$$d(x+1) = (2x+3)d(x) \text{ with } d(0) = 1. \quad (20)$$

A graphical representation of $d(x)$ is shown in figure 6. The graph is continuous and positive on $(0, \infty)$ and, in this interval, possesses continuous derivatives of all orders.

The following properties follow from various substitutions in (17) – (20):

$$d(1/2) = \frac{2\sqrt{2}}{\sqrt{\pi}} \quad (21)$$

$$d(n+1/2) = \frac{2^{(n+3/2)}}{\sqrt{\pi}}(n+1)! \text{ for } n \geq 0 \quad (22)$$

The following result provides an estimate for large values of x :

$$\text{As } x \rightarrow \infty, \quad d(x) \sim \sqrt{2e}(2x)^{x+1}e^{-(x+1)} \quad (23)$$

The $D(x,y)$ function

We introduce a function of two variables $D(x,y)$ defined for positive real values of x and y by

$$D(x,y) = y! \left[\frac{(2x+1)!}{2^x(x)!} \right]^y. \quad (24)$$

so that $d(x)$ can be considered as a special case of $D(x,y)$ when $y = 1$.

$D(x,y)$ satisfies the following properties:

$$D(0,0) = 1 \quad (25)$$

$$D(0,1) = 1 \quad (26)$$

$$D(1,1) = 3 \quad (27)$$

$$D(x,0) = 1 \text{ for } x \geq 0 \quad (28)$$

$$D(0,y) = y! \text{ for } y \geq 0 \quad (29)$$

$$D(x,1) = d(x) \quad (30)$$

$$D(x, (y_1+y_2)) = \frac{(y_1+y_2)}{y_1 y_2} B^{-1}(y_1, y_2) D(x, y_1) D(x, y_2) \quad (31)$$

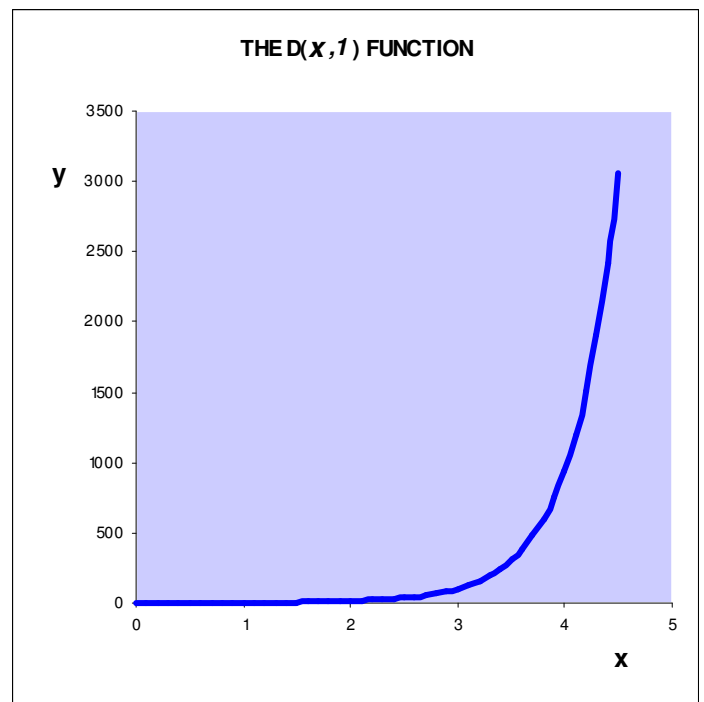


Figure 6. A graphical representation of $D(x,1)$.

Applications of the $D(x,y)$ function

The $D(x,y)$ function provides a convenient means of representing and evaluating the number of Hamilton circuits [3] and Euler circuits [8] in a complete graph K_n .

For $n > 0$, the number of Hamilton circuits in K_n is given by

$$H(n) = D(0,n-1). \quad (32)$$

For n even, the number of Euler circuits in K_n is 0. (33)

For n odd, the number of Euler circuits in K_n is estimated by

$$E(n) = (n-1)D((n-3)/2, n-2) \quad (34)$$

So, for examples, the number of Euler circuits in K_{23} and K_{25} are estimated respectively

$$E(23) = O(10^{234}) \quad (35)$$

$$E(25) = O(10^{288}) \quad (36)$$

Finally, a recurrence relation for $E(n)$ is

$$E(n+2) = (n+1)n^{(n+1)} \left[\frac{(n-2)!}{2^{(n-3)/2} ((n-3)/2)!} \right]^2 E(n) \quad (37)$$

with $E(3) = 2$.

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About the author

John Dwyer is currently *Professor of Computing* at Richmond, The American International University in London. Prior to joining Richmond, he was Deputy Head of the Computer Science Department at the City University of New York. He originally joined the City University of New York as an associate professor of computer science in 1985. He was previously a principal lecturer at Sheffield Hallam University (UK), associate professor at Iona College (US) and senior lecturer at South Bank University (UK). He was also a visiting research fellow at City University in London and lecturer at Westminster University in London.

Professor Dwyer has acted as the external examiner to several PhD examinations and his research interests include the analysis of algorithms, the history of computing and speech processing. He has served as a governor at the Vineyard School in Richmond and he was previously a Fellow of the Institution of Electrical Engineers. Professor Dwyer earned a BSc (Hons) degree and an MSc degree from the University of East Anglia (UK) and a Ph.D. from the University of Essex (UK).

Abstract

In graph theory, the number of circuits in a given graph is closely-related to special functions in mathematics, including the *Gamma function* $\Gamma(x)$ and *Reciprocal Beta function* $B^{-1}(x,y)$. This paper investigates these rapidly-increasing functions and introduces a new function $D(x,y)$, which provides a convenient means of evaluating in a single function the number of Hamilton and Euler circuits in a complete graph K_n .

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